

SURGERY AND HARMONIC SPINORS

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ABSTRACT. Let M be a compact manifold with a fixed spin structure χ . The Atiyah-Singer index theorem implies that for any metric g on M the dimension of the kernel of the Dirac operator is bounded from below by a topological quantity depending only on M and χ . We show that for generic metrics on M this bound is attained.

1. INTRODUCTION

We suppose that M is a compact spin manifold. By a *spin manifold* we will always mean a smooth manifold equipped with an orientation and a spin structure. After choosing a metric g on M , one can define the spinor bundle $\Sigma^g M$ and the Dirac operator $D^g : \Gamma(\Sigma^g M) \rightarrow \Gamma(\Sigma^g M)$ see [6, 12, 8].

Being a self-adjoint elliptic operator D^g shares many properties with the Hodge-Laplacian $\Delta_p^g : \Gamma(\Lambda^p T^* M) \rightarrow \Gamma(\Lambda^p T^* M)$. In particular, if M is compact, then the spectrum is discrete and real, and the kernels of Δ_p^g and D^g are finite-dimensional. Elements of $\ker \Delta_p^g$ resp. $\ker D^g$ are called *harmonic forms* resp. *harmonic spinors*.

However, the relation of Δ_p^g resp. D^g to topology is different. Hodge theory tells us that the Betti numbers $b_p := \dim \ker \Delta_p^g$ only depend on the topological type of M . The dimension of the kernel of D^g is invariant under conformal changes of the metric, however it does depend on the choice of conformal structure. The first examples of this phenomenon were constructed by Hitchin [9], and it was conjectured by several people including Bär and the second named author [2] that $\dim \ker D^g$ depends on the metric for any compact spin manifold of dimension ≥ 3 .

On the other hand, $\dim \ker D^g$ is topologically obstructed. The Index Theorem by Atiyah and Singer gives a topological lower bound on the dimension of the kernel of the Dirac operator. For M a compact spin manifold of dimension n this bound is [12], [2, Section 3]

$$\dim \ker D^g \geq \begin{cases} |\hat{A}(M)|, & \text{if } n \equiv 0 \pmod{4}; \\ 1, & \text{if } n \equiv 1 \pmod{8} \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \pmod{8} \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

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Here the \hat{A} -genus $\hat{A}(M) \in \mathbb{Z}$ and the α -genus $\alpha(M) \in \mathbb{Z}/2\mathbb{Z}$ are invariants of (the spin bordism class of) the differential spin manifold M , and g is any Riemannian metric on M .

It is hence natural to ask whether metrics exist, such that equality holds in (1). Such metrics will be called *D-minimal*. In [13] it is proved that a generic metric on a manifold of dimension ≤ 4 is *D-minimal*. In [2] the same result is proved for manifolds of dimension at least 5 which are simply connected or have certain fundamental groups. The argument in [2] utilizes the surgery-bordism method which has proven itself very powerful in the study of manifolds with positive scalar curvature metrics. In a similar fashion we will use surgery methods to prove the following.

Theorem 1.1. *Let M be a compact connected spin manifold. Then a generic metric on M is *D-minimal*.*

Our method also yields a new proof in dimensions 2, 3 and 4. Since $\dim \ker D$ behaves additively with respect to disjoint union of spin manifolds while the \hat{A} -genus/ α -genus may cancel it is easy to find disconnected manifolds with no *D-minimal* metric.

Let us also mention that if M is a compact Riemann surface of genus ≤ 2 , then *all metrics* are *D-minimal*. The same holds for Riemann surface of genus 3 whose spin structure is not spin bordant 0. However if the genus is ≥ 4 (or equal to 3 with spin structures that are spin bordant 0), then there are also metrics with larger kernel [9], see also [3].

In order to explain the surgery-bordism method in the proof of Theorem 1.1 we have to fix some notation.

A smooth embedding $f : N \rightarrow M$ is called *spin preserving* if the pullback of the orientation and spin structure of M to N under f is the orientation and spin structure of N . If M is a spin manifold we denote by M^- the same manifold with the opposite orientation.

For $l \geq 1$ we denote by $B^l(R)$ the standard l -dimensional open ball of radius R and by $S^{l-1}(R)$ its boundary. We abbreviate $B^l = B^l(1)$ and $S^{l-1} = S^{l-1}(1)$. The standard Riemannian metrics on $B^l(R)$ and $S^{l-1}(R)$ are denoted by g^{flat} and g^{round} . We equip $S^{l-1}(R)$ with the *bounding spin structure*, i.e. the spin structure obtained by restricting the unique spin structure on $B^l(R)$ (if $l > 2$ the spin structure on $S^{l-1}(R)$ is unique, if $l = 2$ it is not).

Let $f : S^k \times \overline{B^{n-k}} \rightarrow M$ be a spin prerserving embedding, Then we define

$$\widetilde{M} = (M \setminus f(S^k \times B^{n-k})) \cup \left(\overline{B^{k+1}} \times S^{n-k-1} \right) / \sim$$

where \sim identifies the boundary of $S^k \times S^{n-k-1}$ with $f(S^k \times S^{n-k-1})$. The topological space \widetilde{M} carries a differential structure and a spin structure such that the inclusions $M \setminus f(S^k \times B^{n-k}) \hookrightarrow \widetilde{M}$ and $\overline{B^{k+1}} \times S^{n-k-1} \hookrightarrow \widetilde{M}$ are spin preserving smooth embeddings.

We say that \widetilde{M} is obtained from M by *surgery of dimension k* or by *surgery of codimension $n - k$* .

The proof of Theorem of Theorem 1.1 relies on the following surgery theorem.

Theorem 1.2. *Let (M, g^M) be a compact n -dimensional Riemannian spin manifold. Let \widetilde{M} be obtained from M by surgery in dimension k , $k \in \{0, 1, \dots, n-2\}$. Then \widetilde{M} carries a metric $g^{\widetilde{M}}$ such that*

$$\dim \ker D^{g^{\widetilde{M}}} \leq \dim \ker D^{g^M}.$$

2. PRELIMINARIES

2.1. Spinor bundles for different metrics. Let M be a spin manifold of dimension n and let g, g' be Riemannian metrics on M . The goal of this paragraph is to identify the spinor bundles of (M, g) and (M, g') using the method of Bourguignon and Gauduchon introduced in [5].

There exists a unique endomorphism $b_{g'}^g$ of TM which is positive, symmetric with respect to g , and satisfies $g(X, Y) = g'(b_{g'}^g X, b_{g'}^g Y)$ for all $X, Y \in TM$. This endomorphism maps g -orthonormal frames at a point to g' -orthonormal frames at the same point and we get a map $b_{g'}^g : \text{SO}(M, g) \rightarrow \text{SO}(M, g')$ of $\text{SO}(n)$ -principal bundles. If we assume that $\text{Spin}(M, g)$ and $\text{Spin}(M, g')$ are equivalent spin structures on M the map $b_{g'}^g$ lifts to a map $\beta_{g'}^g$ of $\text{Spin}(n)$ -principal bundles,

$$\begin{array}{ccc} \text{Spin}(M, g) & \xrightarrow{\beta_{g'}^g} & \text{Spin}(M, g') \\ \downarrow & & \downarrow \\ \text{SO}(M, g) & \xrightarrow{b_{g'}^g} & \text{SO}(M, g') \end{array}.$$

From this we get a map between the spinor bundles $\Sigma^g M$ and $\Sigma^{g'} M$ denoted by the same symbol and defined by

$$\begin{aligned} \Sigma^g M = \text{Spin}(M, g) \times_{\sigma} \Sigma_n &\rightarrow \text{Spin}(M, g') \times_{\sigma} \Sigma_n = \Sigma^{g'} M \\ \psi = [s, \varphi] &\mapsto [\beta_{g'}^g s, \varphi] = \beta_{g'}^g \psi \end{aligned} \quad (2)$$

where (σ, Σ_n) is the complex spinor representation, and where $[s, \varphi]$ denotes the equivalence class of $(s, \varphi) \in \text{Spin}(M, g) \times_{\sigma} \Sigma_n$ for the equivalence relation given by the action of $\text{Spin}(n)$. The map $\beta_{g'}^g$ preserves fiberwise length of spinors.

We define the Dirac operator $D^{g'}$ acting on sections of the spinor bundle for g by

$${}^g D^{g'} = (\beta_{g'}^g)^{-1} \circ D^{g'} \circ \beta_{g'}^g$$

In [5, Thm. 20] the operator ${}^g D^{g'}$ is computed in terms of D^g and some extra terms which are small if g and g' are close. Formulated in a way convenient for us the relationship is

$${}^g D^{g'} \psi = D^g \psi + A_{g'}^g(\nabla^g \psi) + B_{g'}^g(\psi) \quad (3)$$

where $A_{g'}^g \in \text{hom}(T^*M \otimes \Sigma^g M, \Sigma^g M)$ satisfies

$$|A_{g'}^g| \leq C|g - g'|_g \quad (4)$$

and $B_{g'}^g \in \text{hom}(\Sigma^g M, \Sigma^g M)$ satisfies

$$|B_{g'}^g| \leq C(|g - g'|_g + |\nabla^g(g - g')|_g) \quad (5)$$

for some constant C .

In the special case that g' and g are conformal with $g' = F^2 g$ for a positive smooth function F we have

$${}^g D^{g'}(F^{-\frac{n-1}{2}}\psi) = F^{-\frac{n+1}{2}} D^g \psi \quad (6)$$

according to [9, 4, 8].

2.2. Notations for spaces of spinors. Throughout the article φ and ψ and its variants denote spinors, i.e. sections of the spinor bundle. If S is a closed or open subset of M , we write $C^k(S)$ both for the space of k times differentiable functions on S and for the space of k times differentiable spinors. As the bundle will be clear from the context, this will not lead to ambiguities. On $C^k(S)$ we define the norm

$$\|\varphi\|_{C^k(S)} := \sum_{l=0}^k \sup_{x \in S} |\nabla^l \varphi(x)|.$$

We sometimes write $\|\varphi\|_{C^k(S,g)}$ instead of $\|\varphi\|_{C^k(S)}$ to indicate that the spinor bundle and the norm depend on g . The analogous notation is used for Schauder spaces $C^{k,\alpha}$.

Similarly $L^2(S) = L^2(S, g)$ and $H_k^2(S) = H_k^2(S, g)$ denote the space of L^2 -spinors and H_k^2 -spinors. These spaces come with the norms

$$\|\varphi\|_{L^2(S,g)}^2 := \int_S |\varphi|^2 dv^g \quad \|\varphi\|_{H_k^2(S,g)}^2 := \sum_{l=0}^k \int_S |\nabla^l \varphi|^2 dv^g.$$

Let U be an open set. The set of locally C^1 -spinors $C_{\text{loc}}^1(U)$ carries a topology such that $\varphi_i \rightarrow \varphi$ in $C_{\text{loc}}^1(U)$ if and only if $\varphi_i \rightarrow \varphi$ in $C^1(K)$ for any compact subset $K \subset U$.

2.3. Regularity and elliptic estimates. In the following section M is not necessarily compact.

Lemma 2.1. *Let (M, g) be a Riemannian manifold, and let ψ be a spinor of regularity L^2 . If ψ is weakly harmonic, i.e.*

$$\int_M \langle \psi, D\varphi \rangle dv^g = 0$$

for all compactly supported smooth spinors φ , then ψ is smooth.

Lemma 2.2. *Let (M, g) be a Riemannian manifold and let $K \subset M$ a compact subset. Then there is a constant $C = C(K, M, g)$ such that*

$$\|\psi\|_{C^2(K,g)} \leq C \|\psi\|_{L^2(M,g)}$$

for all harmonic spinors ψ on (M, g) .

Proof of the lemmata.

The condition of the first lemma implies $\int \langle \psi, D^2 \Phi \rangle dv^g = 0$ for any compactly supported smooth spinor Φ . Writing down the equation in local coordinates, one can use standard tools from partial differential equations (as for example [7, Theorem 8.13]) to derive via recursion that ψ is contained in $H_k^2(K_1)$ for any $k \in \mathbb{N}$ and any K_1 compact in M , and that

$$\|\psi\|_{H_k^2(K_1,g)} \leq C \|\psi\|_{L^2(M,g)}. \quad (7)$$

Suppose that the boundary of K_1 is smooth. One then uses the Sobolev embedding $H_k^2(K_1, g) \rightarrow C^1(K_1, g)$ for $k > n/2 + 1$ (see [1, Theorem 6.2]), and we get $\psi \in C^1(K_1, g)$ and an estimate for $\|\psi\|_{C^1(K_1, g)}$ analogous to (7). Now one can use Schauder estimates as in [7, Theorem 6.6] to conclude that ψ is smooth on any compactum K contained in the interior of K_1 , and in order to derive a C^2 estimate. \square

Lemma 2.3 (Ascoli's theorem, [1, Theorem 1.30 and 1.31]). *Let φ_i be a sequence bounded in $C^{1,\alpha}(K)$. Then a subsequence converges in $C^1(K)$.*

2.4. Removal of singularities lemma. In the proof of Theorem 1.2 we will need the following lemma.

Lemma 2.4. *Let (M, g) be an n -dimensional Riemannian spin manifold and let $S \subset M$ be a compact submanifold of dimension $k \leq n - 2$. Assume that φ is a spinor field such that $\|\varphi\|_{L^2(M)} < \infty$ and $D^g\varphi = 0$ weakly on $M \setminus S$. Then $D^g\varphi = 0$ holds weakly also on M .*

Proof. Let ψ be a smooth spinor compactly supported in M . We have to show that

$$\int_M \langle \varphi, D^g\psi \rangle dv^g = 0. \quad (8)$$

Let $U_S(\varepsilon)$ be the set of points of distance at most ε to S . For a small $\varepsilon > 0$ we choose a smooth function $\eta : M \rightarrow [0, 1]$ such that $\eta = 1$ on $U_S(\varepsilon)$, $|\text{grad}\eta| \leq 2/\varepsilon$ and $\eta = 0$ outside $U_S(2\varepsilon)$. We rewrite the left hand side of (8) as

$$\begin{aligned} \int_M \langle \varphi, D^g\psi \rangle dv^g &= \int_M \langle \varphi, D^g((1 - \eta)\psi + \eta\psi) \rangle dv^g \\ &= \int_M \langle \varphi, D^g((1 - \eta)\psi) \rangle dv^g \\ &\quad + \int_M \langle \varphi, \eta D^g\psi \rangle dv^g + \int_M \langle \varphi, \text{grad}\eta \cdot \psi \rangle dv^g. \end{aligned}$$

As $D^g\varphi = 0$ weakly on $M \setminus S$ the first term vanishes. The absolute value of the second term is bounded by

$$\|\varphi\|_{L^2(U_S(2\varepsilon))} \|D^g\psi\|_{L^2(U_S(2\varepsilon))}$$

which tends to 0 as $\varepsilon \rightarrow 0$. Finally, the absolute value of the third term is bounded by

$$\begin{aligned} \frac{2}{\varepsilon} \|\varphi\|_{L^2(U_S(2\varepsilon))} \|\psi\|_{L^2(U_S(2\varepsilon))} &\leq \frac{C}{\varepsilon} \|\varphi\|_{L^2(U_S(2\varepsilon))} (\text{Vol}(U_S(2\varepsilon) \cap \text{supp}(\psi)))^{\frac{1}{2}} \\ &\leq C \|\varphi\|_{L^2(U_S(2\varepsilon))} \varepsilon^{\frac{n-k}{2}-1}. \end{aligned}$$

Since $n - k \geq 2$, the third term also tends to 0 as $\varepsilon \rightarrow 0$. \square

2.5. Products with spheres. The spectrum of $(D^{g^{\text{round}}})^2$ is bounded from below by $l^2/4$.

If (M, g) and (N, h) are compact Riemannian spin manifolds then the squared Dirac operator $(D^{g+h})^2$ on $(M \times N, g + h)$ can be identified with $(D^g)^2 + (D^h)^2$. We conclude the following.

Proposition 2.5. *Let (M, g) be a compact spin manifold and $l \geq 1$. Then the spectrum of $(D^{g+g^{\text{round}}})^2$ on $M \times S^l$ is bounded from below by $l^2/4$.*

3. PROOF OF THEOREM 1.2

Our standing assumptions are: (M, g) is a compact Riemannian spin manifold of dimension n together with a k -dimensional submanifold S of M diffeomorphic to S^k . We assume $n - k \geq 2$. The restriction of g to S is denoted by h . Let $\nu \rightarrow S$ be the normal bundle of S . We assume furthermore that a trivialization of the normal bundle is given, that is a vector bundle map $\iota : \mathbb{R}^{n-k} \times S \rightarrow \nu$. We assume that ι is fiberwise an isometry.

For $R > 0$ we denote by $\nu(R)$ the disk bundle of vectors of length $\leq R$ in ν . For sufficiently small R the normal exponential map \exp^ν of S defines a diffeomorphism of $\nu(R)$ onto a neighborhood of S . For such small $R > 0$ one has

$$U_S(R) = (\exp^\nu \circ \iota)(\overline{B^{n-k}(R)} \times S) = \exp^\nu(\nu(R)).$$

Lemma 3.1. *Let $n \geq 3$. Let χ be the canonical spin structure on \mathbb{R}^{n-1} , let χ_b be the bounding spin structure on S^1 and χ_{nb} the non-bounding spin structure on S^1 . There is a diffeomorphism from $F : \mathbb{R}^{n-1} \times S^1$ to itself preserving the linear structure of \mathbb{R}^{n-1} with*

$$F^*(\chi \times \chi_b) = \chi \times \chi_{nb}.$$

Proof. Let $\gamma : S^1 \rightarrow \text{SO}(n-1)$ be a generator of $\pi_1(\text{SO}(n-1))$. Then the map $(X, x) \mapsto (\gamma(x)X, x)$ is a diffeomorphism as desired. \square

Let $\exp^\nu : \nu \rightarrow M$ be the restriction of the exponential map to ν . Close to the zero section of ν , \exp^ν is a diffeomorphism onto its image, and hence for small $\varepsilon > 0$ the map

$$I_\iota : \mathbb{R}^{n-k} \times S, \quad (X, x) \mapsto \exp \left(R \frac{\iota(X, x)}{\sqrt{1 + \|X\|^2}} \right)$$

is a diffeomorphism onto the interior of $U_S(R)$. The spin structure on M induces a spin structure on $\mathbb{R}^{n-k} \times S$. If $k \geq 2$, then the spin structure on $\mathbb{R}^{n-k} \times S$ is unique. However, in the case $k = 1$, the induced spin structure might be $\chi \times \chi_b$ or $\chi \times \chi_{nb}$. If the induced spin structure is $\chi \times \chi_{nb}$, we replace ι by $\iota' = \iota \circ F$, and the spin structure induced by $I_{\iota'}$ is $\chi \times \chi_b$. Hence, we can assume from now on without loss of generality that the trivialization ι induces the spin structure $\chi \times \chi_b$.

3.1. Approximation by a metric of product form near S . In the following $r(x)$ denotes the distance from the point x to S with respect to the metric g .

Lemma 3.2. *For sufficiently small $R > 0$ there is a constant $C > 0$ so that*

$$G = g - ((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)$$

satisfies

$$|G(x)| \leq Cr(x), \quad |\nabla G(x)| \leq C$$

on $U_S(R)$.

Note that in this lemma the function $r(x)$ is by definition the distance of x to S with respect to g but it coincides with the distance of x to S with respect to the metric $((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)$

Proof. Since $x \mapsto \nabla G(x)$ is continuous on a neighborhood of S we can find a constant C such that $|\nabla G(x)| \leq C$ for sufficiently small $R > 0$. Now, let $x \in S$. At first the spaces $T_x S$ and ν_x are orthogonal with respect to the two scalar products $g(x)$ and $((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)(x)$. It is also clear that these two scalar products coincide on $T_x S$. Since the differential $d(\exp^\nu \circ \iota)$ is an isometry, they coincide also on ν_x . This implies that $g(x) = ((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)(x)$ and hence that $G(x) = 0$. We obtain that G vanishes on S . Since G is C^1 , $|G|$ is 1-lipschitzian and thus there exists $C > 0$ such that $|G(x)| \leq Cr(x)$. \square

The following proposition allows us to assume that the metric g has product form close to the surgery sphere S .

Proposition 3.3. *Let (M, g) and S be as above. Then there is a metric \tilde{g} on M and $\varepsilon > 0$ such that $d^g(x, S) = d^{\tilde{g}}(x, S)$, \tilde{g} has product form on $U_S(\varepsilon)$ and*

$$\dim \ker D^{\tilde{g}} \leq \dim \ker D^g.$$

For $\delta > 0$ let η be a smooth cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $U_S(\delta)$, $\eta = 0$ on $M \setminus U_S(2\delta)$, and $|d\eta|_g \leq 2/\delta$. We set

$$g_\delta = \eta((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h) + (1 - \eta)g.$$

Then $d^g(x, S) = d^{g_\delta}(x, S) = r(x)$. Through a series of lemmas we will prove the proposition for $\tilde{g} = g_\delta$ for δ sufficiently small.

In the following estimates C denotes a constant whose values might vary from one line to another, which is independent of δ and η but might depend on M, g, S . Terms denoted by $o_i(1)$ tend to zero when $i \rightarrow \infty$.

Lemma 3.4. *Let δ_i be a sequence with $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. Let φ_i be a sequence of spinors on (M, g_{δ_i}) such that $D^{g_{\delta_i}} \varphi_i = 0$ and $\int_M |\varphi_i|^2 dv^{g_{\delta_i}} = 1$. Then the sequence $\beta_g^{g_{\delta_i}} \varphi_i$ is bounded in $H_1^2(M, g)$.*

Proof. As $\int |\beta_g^{g_{\delta_i}} \varphi_i|^2 dv^g = 1 + o_i(1)$ we have to show that $\alpha_i = \sqrt{\int_M |\nabla^g(\beta_g^{g_{\delta_i}} \varphi_i)|_g^2 dv^g}$ is bounded. We assume the opposite, that is $\alpha_i \rightarrow \infty$, and set $\psi_i = \alpha_i^{-1} \beta_g^{g_{\delta_i}} \varphi_i$. Then we have ${}^g D^{g_{\delta_i}} \psi_i = 0$ since $\beta_{g_{\delta_i}}^g \circ \beta_g^{g_{\delta_i}} = \text{Id}$, so formula (3) gives us

$$\begin{aligned} 1 &= \int_M |\nabla^g \psi_i|_g^2 dv^g \\ &= \int_M (|D^g \psi_i|^2 - \frac{1}{4} \text{scal}^g |\psi_i|^2) dv^g \\ &= \int_M (|A_{g_{\delta_i}}^g(\nabla^g \psi_i) + B_{g_{\delta_i}}^g(\psi_i)|^2 - \frac{1}{4} \text{scal}^g |\psi_i|^2) dv^g \\ &\leq \int_M (2|A_{g_{\delta_i}}^g(\nabla^g \psi_i)|^2 + 2|B_{g_{\delta_i}}^g(\psi_i)|^2 - \frac{1}{4} \text{scal}^g |\psi_i|^2) dv^g. \end{aligned}$$

Using (4), (5), Lemma 3.2, and the fact that g and g_{δ_i} coincide outside $U_S(2\delta_i)$ we get

$$\begin{aligned} 1 &\leq C\delta_i^2 \int_{U_S(2\delta_i)} |\nabla^g \psi_i|_g^2 dv^g + C \int_{U_S(2\delta_i)} |\psi_i|^2 dv^g + C \int_M |\psi_i|^2 dv^g \\ &\leq C\delta_i^2 + C \int_{U_S(2\delta_i)} |\psi_i|^2 dv^g + \alpha_i^{-2}(1 + o_i(1)) \\ &\leq C \int_{U_S(2\delta_i)} |\psi_i|^2 dv^g + o_i(1) \end{aligned}$$

As ψ_i is bounded in $H_1^2(M, g)$, a subsequence converges weakly in $H_1^2(M, g)$ and strongly in $L^2(M, g)$ to a limit spinor $\psi \in H_1^2(M, g)$. Hence for this subsequence

$$\int_{U_S(2\delta_i)} |\psi_i|_g^2 dv^g \rightarrow 0$$

which implies a contradiction. \square

Lemma 3.5. *Again let δ_i be a sequence with $\delta_i \rightarrow 0$ as $i \rightarrow \infty$ and let φ_i be a sequence of spinors on (M, g_{δ_i}) such that $D^{g_{\delta_i}} \varphi_i = 0$ and $\int_M |\varphi_i|^2 dv^{g_{\delta_i}} = 1$. Then, after passing to a subsequence, $\beta_g^{g_{\delta_i}} \varphi_i$ converges weakly in $H_1^2(M, g)$ and strongly in $L^2(M, g)$ to a harmonic spinor on (M, g) .*

Proof. According to the previous Lemma the sequence $\beta_g^{g_{\delta_i}} \varphi_i$ is bounded in $H_1^2(M, g)$ and hence a subsequence converges weakly in $H_1^2(M, g)$. After passing to a subsequence once again we obtain strong convergence in $L^2(M, g)$. Denote the limit spinor by φ .

For any $\varepsilon > 0$ Lemma 2.2 implies that $\beta_g^{g_{\delta_i}} \varphi_i$ is bounded in $C^2(M \setminus U_S(\varepsilon))$, and Lemma 2.3 then implies that a subsequence converges in $C^1(M \setminus U_S(\varepsilon))$. Hence the limit φ is in $C_{\text{loc}}^1(M \setminus S)$ and satisfies $D^g \varphi = 0$ on $M \setminus U(S)$. Since φ is in $L^2(M, g)$ it follows from Lemma 2.4 that φ is a weak solution of $D\psi = 0$ on (M, g) . By elliptic regularity theory φ is a strong solution and a harmonic spinor on (M, g) . \square

Proof of Proposition 3.3. Let $m = \liminf_{\delta \rightarrow 0} \dim \ker D^{g_\delta}$. For sufficiently small δ let $\varphi_\delta^1, \dots, \varphi_\delta^m \in \ker D^{g_\delta}$ be spinors such that

$$\int_M \langle \varphi_\delta^j, \varphi_\delta^k \rangle dv^{g_\delta} = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases} \quad (9)$$

According to Lemma 3.5 there are spinors $\varphi^1, \dots, \varphi^m \in \ker D^g$ and a sequence $\delta_i \rightarrow 0$ such that $\beta_g^{g_{\delta_i}} \varphi_\delta^j$ converges to φ^j weakly in $H_1^2(M, g)$ and strongly in $L^2(M, g)$ for $j = 1, \dots, m$. Because of strong L^2 -convergence the orthogonality relation (9) is preserved in the limit so $\dim \ker D^g \geq m$. Hence there is a $\delta_0 > 0$ so that $\dim \ker D^{g_{\delta_0}} = m \leq \dim \ker D^g$ and the Proposition is proved with $\tilde{g} = g_{\delta_0}$. \square

3.2. Proof for metrics of product form near S . We assume that g is a product metric on $U_S(R_{\max})$ for some $R_{\max} > 0$, as we may from Proposition 3.3. In polar coordinates $(r, \Theta) \in (0, R_{\max}) \times S^{n-k-1}$ on $B^{n-k}(R_{\max})$ we get

$$g = g^{\text{flat}} + h = dr^2 + r^2 g^{\text{round}} + h.$$

$$0 < \rho \ll r_0 < r_1/2 \ll R_{\max}$$

FIGURE 1. Hierachy of variables

Let $\rho > 0$ be a small number which we will finally let tend to 0 (see also Figure 3.2). We decompose M into three parts

- (1) $M \setminus U_S(R_{\max})$,
- (2) $(\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$,
- (3) $U_S(\rho/2) = B^{n-k}(\rho/2) \times S^k$.

The manifold \widetilde{M} is obtained by removing part (3) and by gluing in $S^{n-k-1} \times B^{k+1}$, that is \widetilde{M} is the union of

- (1) $M \setminus U_S(R_{\max})$,
- (2) $(\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$,
- (3') $S^{n-k-1} \times B^{k+1}$.

We now define a sequence of metrics g_ρ on \widetilde{M} such that the theorem holds for small $\rho > 0$. The metrics g_ρ will coincide with g on part (1), but will be modified in part (2) in order to close up nicely in part (3').

Let r_0, r_1 be fixed such that $2\rho < r_0 < r_1/2 < R_{\max}/2$. Define g_ρ on \widetilde{M} by

- (1) $g_\rho = g$ on $M \setminus U_S(R_{\max})$,
- (2) $g_\rho = F^2(dr^2 + r^2 g^{\text{round}} + f_\rho^2 h)$ on $(\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$, where F and f_ρ satisfy

$$F(r) = \begin{cases} 1, & \text{if } r_1 < r < R_{\max}; \\ 1/r, & \text{if } r < r_0, \end{cases} \quad \text{and} \quad f_\rho(r) = \begin{cases} 1, & \text{if } r > 2\rho; \\ r, & \text{if } r < \rho. \end{cases}$$

- (3') $g_\rho = g^{\text{round}} + \gamma_\rho$ on $S^{n-k-1} \times B^{k+1}$ where γ_ρ is some metric so that g_ρ is smooth.

The metric g_ρ is visualized in Figure 3.2. In order to visualize the metric g_ρ two projections are drawn. In both projections the horizontal direction represents $-\log r$. In the first projection the vertical direction indicates the size of the co-sphere S^{n-k-1} . In the second projection the vertical direction indicates the size of S which is fiberwise homothetic to $(S \cong S^k, h)$.

We are now going to prove that

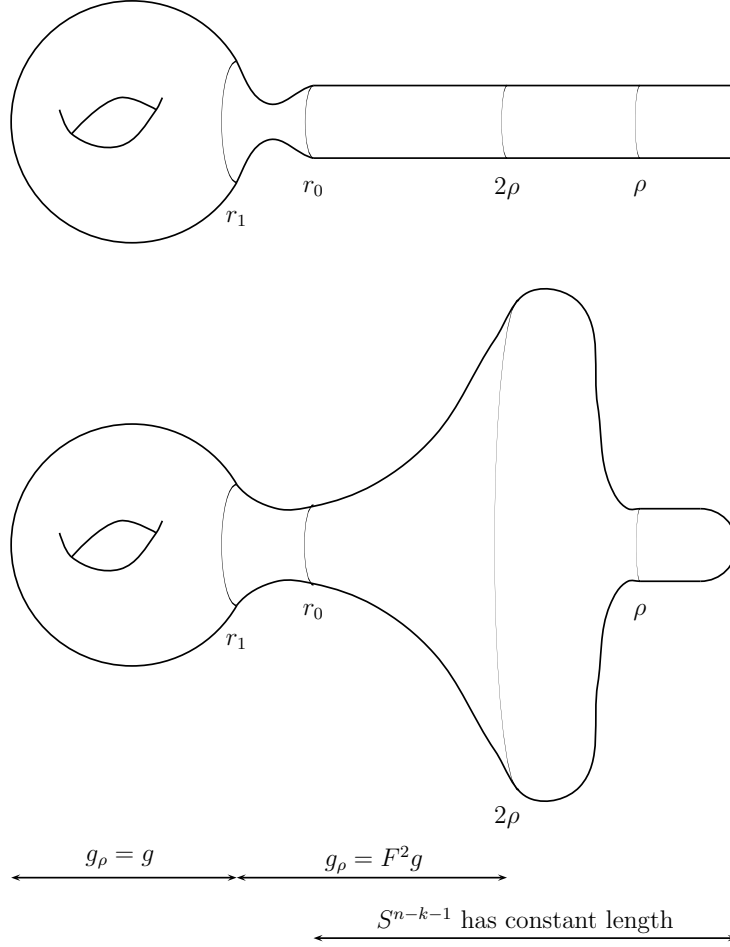
$$\dim \ker D^{g_\rho} \leq \dim \ker D^g \tag{10}$$

for small $\rho > 0$. Before proving (10), we need some estimates.

For $\alpha \in (0, \rho/2)$, let $\widetilde{U}(\alpha) = \widetilde{M} \setminus (M \setminus U_S(\alpha))$ so that $M \setminus U_S(\alpha) = \widetilde{M} \setminus \widetilde{U}(\alpha)$.

Proposition 3.6. *Let $s \in (0, r_1/2)$. Let ψ_ρ be a harmonic spinor on (\widetilde{M}, g_ρ) . Then for $\rho \in (0, s)$ it holds that*

$$\frac{(n-k-1)^2}{32} \int_{\widetilde{U}(s) \setminus \widetilde{U}(2\rho)} |F^{\frac{n-1}{2}} \psi_\rho|^2 dv^g \leq \int_{\widetilde{U}(2s) \setminus \widetilde{U}(s)} |F^{\frac{n-1}{2}} \psi_\rho|^2 dv^g.$$

FIGURE 2. The metric g_ρ .

Proof. Let $\eta \in C^\infty(\widetilde{M})$ be a cut-off function with $0 \leq \eta \leq 1$, $\eta = 1$ on $\widetilde{U}(s)$, $\eta = 0$ on $\widetilde{M} \setminus \widetilde{U}(2s)$, and

$$|d\eta|_g \leq \frac{2}{s}. \quad (11)$$

The spinor $\eta\psi_\rho$ is compactly supported in $\widetilde{U}(2s)$. Moreover, the metric g_ρ can be written as $g_\rho = g^{\text{round}} + h_\rho$ on $\widetilde{U}(2s)$ where the metric h_ρ is equal to $r^{-2}dr^2 + r^{-2}f_\rho^2 h$ on $\widetilde{U}(2s) \setminus \widetilde{U}(\rho/2)$ and is equal to γ_ρ on $S^{n-k-1} \times B^{k+1} = \widetilde{U}(\rho/2)$. Hence $(\widetilde{U}(2s), g_\rho)$ is isometric to an open subset of a manifold of the form $S^{n-k-1} \times N$ equipped with a product metric $g^{\text{round}} + g_N$, where N is compact. By Proposition 2.5 the squared eigenvalues of the Dirac operator on this product manifold are greater than or equal

to $(n - k - 1)^2/4$. Writing the Rayleigh quotient of $\eta\psi_\rho$ we obtain

$$\frac{(n - k - 1)^2}{4} \leq \frac{\int_{\tilde{U}(2s)} |D^{g_\rho}(\eta\psi_\rho)|^2 dv^{g_\rho}}{\int_{\tilde{U}(2s)} |\eta\psi_\rho|^2 dv^{g_\rho}}. \quad (12)$$

Since $D^{g_\rho}\psi_\rho = 0$ we have $D^{g_\rho}(\eta\psi_\rho) = \text{grad}^{g_\rho}\eta \cdot \psi_\rho$ so

$$|D^{g_\rho}(\eta\psi_\rho)|^2 = |\text{grad}^{g_\rho}\eta \cdot \psi_\rho|^2 = |d\eta|_{g_\rho}^2 |\psi_\rho|_{g_\rho}^2. \quad (13)$$

By definition $d\eta$ is supported in $\tilde{U}(2s) \setminus \tilde{U}(s)$. On $\tilde{M} \setminus \tilde{U}(2\rho)$ we have $g_\rho = F^2g$. Moreover, by Relation (11) and since $F = 1/r$ on the support of $d\eta$, we have

$$|d\eta|_{g_\rho}^2 = r^2 |d\eta|_g^2 \leq \frac{4r^2}{s^2}$$

and hence

$$|D^{g_\rho}(\eta\psi_\rho)|^2 \leq \frac{4r^2}{s^2} |\psi_\rho|^2,$$

Since $g_\rho = r^{-2}g$ on $\tilde{U}(2s) \setminus \tilde{U}(s)$ we have $dv^{g_\rho} = r^{-n} dv^g$. Using equation (13) it follows that

$$\begin{aligned} \int_{\tilde{U}(2s)} |D^{g_\rho}(\eta\psi_\rho)|^2 dv^{g_\rho} &\leq \frac{4}{s^2} \int_{\tilde{U}(2s) \setminus \tilde{U}(s)} r^{2+(n-1)-n} |r^{-\frac{n-1}{2}} \psi_\rho|^2 dv^g \\ &\leq \frac{8}{s} \int_{\tilde{U}(2s) \setminus \tilde{U}(s)} |F^{\frac{n-1}{2}} \psi_\rho|^2 dv^g, \end{aligned} \quad (14)$$

where we also use that $r \leq 2s$ on the domain of integration. Since $\eta \in [0, 1]$ on $\tilde{U}(2s) \setminus \tilde{U}(s)$, since $\eta = 1$ on $\tilde{U}(s)$ and since $g_\rho = r^{-2}g$ on $\tilde{U}(s) \setminus \tilde{U}(2\rho)$, we have

$$\begin{aligned} \int_{\tilde{U}(2s)} |\eta\psi_\rho|^2 dv^{g_\rho} &\geq \int_{\tilde{U}(s) \setminus \tilde{U}(2\rho)} |\psi_\rho|^2 dv^{g_\rho} \\ &= \int_{\tilde{U}(s) \setminus \tilde{U}(2\rho)} r^{(n-1)-n} |r^{-\frac{n-1}{2}} \psi_\rho|^2 dv^g \\ &\geq \frac{1}{s} \int_{\tilde{U}(s) \setminus \tilde{U}(2\rho)} |F^{\frac{n-1}{2}} \psi_\rho|_g^2 dv^g, \end{aligned} \quad (15)$$

where we use that $r \leq s$ in the last inequality. Plugging (14) and (15) into (12) we get

$$\frac{(n - k - 1)^2}{4} \leq \frac{\frac{8}{s} \int_{\tilde{U}(2s) \setminus \tilde{U}(s)} |F^{\frac{n-1}{2}} \psi_\rho|^2 dv^g}{\frac{1}{s} \int_{\tilde{U}(s) \setminus \tilde{U}(2\rho)} |F^{\frac{n-1}{2}} \psi_\rho|_g^2 dv^g}$$

and hence Proposition 3.6 follows. \square

Proof of Theorem 1.2. As explained above we need to prove Relation (10), for a contradiction assume that it is false. Then there is a strictly decreasing sequence $\rho_i \rightarrow 0$ such that $\dim \ker D^g < \dim \ker D^{g_{\rho_i}}$ for all i . To simplify the notation for subsequences we define $E = \{\rho_i : i \in \mathbb{N}\}$. We have $0 \in \overline{E}$ and passing to a subsequence of ρ_i means passing to a subset $E' \subset E$ of with $0 \in \overline{E'}$.

Let $m = \dim \ker D^g + 1$. For all $\rho \in E$ we can find D^{g_ρ} -harmonic spinors $\psi_\rho^1, \dots, \psi_\rho^m$ on (\tilde{M}, g_ρ) such that

$$\int_{\tilde{M} \setminus \tilde{U}(s)} \langle \psi_\rho^j, \psi_\rho^k \rangle dv^g = \int_{\tilde{M} \setminus \tilde{U}(s)} \langle \psi_\rho^j, \psi_\rho^k \rangle dv^g = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k, \end{cases} \quad (16)$$

where $s \leq r_0 < r_1/2$ is fixed as above. Let $\varphi_\rho^j = F^{\frac{n-1}{2}} \psi_\rho^j$. These spinor fields are defined on $M \setminus U(2\rho)$ and by (6) they are D^g -harmonic.

Step 1. Let $\delta \in (0, R_{\max})$. For $\rho > 0$ small enough we have

$$\int_{M \setminus U(\delta)} |\varphi_\rho^j|^2 dv^g \leq \frac{(n-k-1)^2 + 32}{(n-k-1)^2}. \quad (17)$$

By Proposition 3.6 we have

$$\int_{U(s) \setminus U(2\rho)} |\varphi_\rho^j|^2 dv^g \leq \frac{32}{(n-k-1)^2} \int_{U(2s) \setminus U(s)} |\varphi_\rho^j|^2 dv^g.$$

and hence if $2\rho \leq \delta$ it follows that

$$\int_{U(s) \setminus U(\delta)} |\varphi_\rho^j|^2 dv^g \leq \frac{32}{(n-k-1)^2} \int_{M \setminus U(s)} |\varphi_\rho^j|^2 dv^g.$$

It follows that

$$\begin{aligned} \int_{M \setminus U(\delta)} |\varphi_\rho^j|^2 dv^g &= \int_{M \setminus U(s)} |\varphi_\rho^j|^2 dv^g + \int_{U(s) \setminus U(\delta)} |\varphi_\rho^j|^2 dv^g \\ &\leq \left(1 + \frac{32}{(n-k-1)^2}\right) \int_{M \setminus U(s)} |\varphi_\rho^j|^2 dv^g. \end{aligned}$$

From (16) we now obtain Inequality (17).

Step 2. There exists $E' \subset E$ with $0 \in \overline{E'}$ and spinors $\Phi^1, \dots, \Phi^m \in C^1(M \setminus S)$, D^g -harmonic on $(M \setminus S, g)$ such that φ_ρ^j tend to Φ^j in $C_{\text{loc}}^1(M \setminus S)$ as $\rho \rightarrow 0$, $\rho \in E'$.

Let $Z \in \mathbb{N}$ be an integer, $Z > 1/s$. By (17) the sequence $\{\varphi_\rho^j\}_{\rho \in E}$ is bounded in $L^2(M \setminus U(1/Z))$. By Lemma 2.2 it follows that $\{\varphi_\rho^j\}_{\rho \in E}$ is bounded in $C^2(M \setminus U(2/Z))$ for all sufficiently large Z . For a fixed $Z_0 > 1/s$ we apply Lemma 2.3 and conclude that for any j there is a subsequence $\{\varphi_\rho^j\}_{\rho \in E_0}$ of $\{\varphi_\rho^j\}_{\rho \in E}$ that converges in $C^1(M \setminus U(2/Z_0))$ to a spinor Φ_0^j . Similarly we construct further and further subsequences $\{\varphi_\rho^j\}_{\rho \in E_i}$ converging to Φ_i^j in $C^1(M \setminus U(2/(Z_0 + i)))$ with $E_i \subset E_{i-1} \subset \dots \subset E_0 \subset E$, $0 \in \overline{E_i}$. Obviously Φ_i^j extends Φ_{i-1}^j . Define $E' \subset E$ as consisting of one ρ_i from each E_i chosen so that $\rho_i \rightarrow 0$ as $i \rightarrow \infty$. Then the sequence $\{\varphi_\rho^j\}_{\rho \in E'}$ converges in $C_{\text{loc}}^1(M \setminus S)$ to a spinor Φ^j . As ψ_ρ^j is D^g -harmonic on $(M \setminus U(2\rho))$ the $C_{\text{loc}}^1(M \setminus S)$ -convergence implies that $D^g \Phi^j = 0$ on $M \setminus S$. We have proved Step 2.

Step 3. Conclusion.

Let $j \in \{1, \dots, m\}$. By (17) we conclude that

$$\int_{M \setminus S} |\Phi^j|^2 dv^g \leq \frac{(n-k-1)^2 + 32}{(n-k-1)^2}$$

and hence $\Phi^j \in L^2(M)$. By Lemma 2.4 and elliptic regularity Φ^j is harmonic and smooth on all of (M, g) . Since $M \setminus U(s)$ is a relatively compact subset of $M \setminus S$ the normalization (16) is preserved in the limit $\rho \rightarrow 0$ and hence

$$\int_{M \setminus U(s)} \langle \Phi^j, \Phi^k \rangle dv^g = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

This proves that Φ^1, \dots, Φ^m are linearly independent harmonic spinors on (M, g) and hence $\dim \ker D^g \geq m$ which contradicts the definition of m . This proves Relation (10) and Theorem 1.2. \square

4. PROOF OF THEOREM 1.1

The proof will follow the argument of [2] so we introduce notation in accordance to that paper. For a compact spin manifold M the space of smooth Riemannian metrics on M is denoted by $\mathcal{R}(M)$ and the subset of D -minimal metrics is denoted by $\mathcal{R}_{\min}(M)$.

From standard results in perturbation theory it follows that $\mathcal{R}_{\min}(M)$ is open in the C^1 -topology on $\mathcal{R}(M)$ and if $\mathcal{R}_{\min}(M)$ is not empty then it is dense in $\mathcal{R}(M)$ in all C^k -topologies, $k \geq 1$, see for example [13, Prop. 3.1]. We define the word generic to mean these open and dense properties satisfied by $\mathcal{R}_{\min}(M)$ if non-empty. Theorem 1.1 is then equivalent to the following.

Theorem 4.1. *Let M be a compact connected spin manifold. Then there is a D -minimal metric on M .*

Before we start the proof we note the following consequence of Theorem 1.2.

Proposition 4.2. *Let N be a compact spin manifold which has a D -minimal metric and suppose that M is obtained from N by surgery of codimension ≥ 2 . Then M has a D -minimal metric.*

Proof. This follows from Theorem 1.2 since the left hand side of (1) is the same for M and N while the right hand side may only decrease. \square

From the proof of handle decompositions of bordisms we have the following.

Proposition 4.3. *Suppose that M is connected, $\dim M \geq 3$, and that M is spin bordant to a manifold N . Then M can be obtained from N by a sequence of surgeries of codimension ≥ 2 .*

Proof. The statement follows from [10, VII Theorem 3] if $\dim M = 3$. If $\dim M \geq 4$, then we can do surgery in dimension 0 and 1 at a given spin cobordism between M and N , and obtain a connected, simply connected spin cobordism W between M and N . It then follows from [11, VIII 3.1] that one can obtain M from N by surgeries of dimension $0, \dots, n - 2$. \square

Proof of Theorem 4.1. From the solution of the Gromov-Lawson conjecture by Stolz [14] together with knowledge of some explicit manifolds with D -minimal metrics one can show that any compact spin manifold is spin bordant to a manifold with a D -minimal metric, this is worked out in detail in [2, Prop. 3.9]. We may thus assume that the given manifold M is spin bordant to a manifold N equipped with a D -minimal metric. The Theorem now follows from Propositions 4.2 and 4.3 if $\dim M \geq 3$.

Now, let $\dim M = 2$. If $\alpha(M) = 0$, then M can be obtained by adding handles to S^2 , i.e. by 0-dimensional surgery. If $\alpha(M) \neq 0$, then M can be obtained by adding handles to T^2 where T^2 carries the spin structure with $\alpha \neq 0$. Any metric on T^2

with that spin structure has a 2-dimensional kernel, and is thus D -minimal. With Proposition 4.2 we get Theorem 4.1 in the 2-dimensional case. \square

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